THE SPECTRUM OF SMALL PERTURBATIONS OF MOTION OF A FLUID BETWEEN ROTATING SPHERICAL SURFACES

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Until recently open flows were almost exclusively investigated in the theory of hydrodynamic stability. As regards closed flows, the motion of a fluid between rotating cylinders (Taylor's problem) had been the subject of the most detailed analysis. That flow is not, however, a typical example of closed flows in which the loss of stability is entirely different [1]. The motion of a fluid between rotating spherical surfaces may be taken as the most typical example. Such motion between an inner sphere of radius r_1, rotating at an angular velocity inside a stationary outer sphere of radius r_2 was analyzed in paper [2].

The behavior of three separate perturbations at small Reynolds numbers and $r_2/r_1 \equiv a = 2$ were investigated. However, the knowledge of a wider spectrum of normal perturbations in the whole range of Reynolds numbers is required for the derivation of the nonlinear theory of stability. In the present paper, expansions of normal perturbations, and of their decrements into power series of R are derived and analyzed. It is shown that both monotonous (with real decrements), and oscillatory (with imaginary decrements) perturbations are possible in this problem. Generation of complex perturbations occurs at a certain critical value $R_* \neq 0$. Decrement corrections were computed at slow rotation by the perturbation method for $a = 2.5$, 2.0, 1.7.

1. **Basic laminar flow.** As units of measurement of the radius, velocity, and pressure we take respectively r_1, ν/r_1, $\rho\nu^2/r_1^2$, where ρ is the density, and ν the kinematic viscosity of the fluid. The equations of stationary motion are

$$(\mathbf{U}\nabla)\,\mathbf{U} = -\nabla P - \operatorname{rot}\operatorname{rot}\mathbf{U}, \qquad \operatorname{div}\mathbf{U} = 0 \tag{1.1}$$

with boundary conditions ($\mathbf{r}_0$ is the unit vector)

$$\mathbf{U}|_{s_1} = R\mathbf{n}\times\mathbf{r}_0, \qquad \mathbf{U}|_{s_2} = 0 \quad \left(R = \frac{r_1^2\Omega}{\nu}\right) \tag{1.2}$$

Here, R is the Reynolds number. For small Reynolds numbers the solution may be sought in the form of series [3]

$$\mathbf{U} = R\mathbf{U}_1 + R^2\mathbf{U}_2 + \ldots, \qquad P = RP_1 + R^2P_2 + \ldots \tag{1.3}$$

Function $\mathbf{U}_1$ is given in [4], while $\mathbf{U}_2$ was found by Bratukhin [2] with the use of spherical vector functions [5].

$$\mathbf{U}_1 = \varphi(r)\,\mathbf{n}\times\mathbf{r}, \qquad \varphi(r) = \frac{1}{a^3-1}\left(\frac{a^3}{r^3}-1\right)$$

$$\mathbf{U}_2 = F(r)\,\mathbf{r}_0 Y_2 + G(r)\,r\nabla Y_2 \qquad (Y_2 = 1/2\,(3\cos^2\vartheta - 1)) \tag{1.4}$$

$$F = \frac{a^6}{2(a^3-1)^2}\frac{1}{r^4}\left[-\frac{c_2}{2} + r + c_4 r^2 + \frac{r^4}{a^3} + \frac{c_3 r^5}{3} + \frac{c_1}{5}r^7\right], \quad G = \frac{1}{6r}(r^2F)'$$

Coefficients c_1, c_2, c_3, c_4 are defined by the boundary conditions (1.2). Streamlines of the secondary flow are given in [2]. For $a = 2$ the amplitude of $\mathbf{U}_2$ is smaller than that of $\mathbf{U}_1$ by approximately two orders of magnitude. This ratio increases when $a \to 1$, and decreases for higher values of a.

2. Perturbations in a stationary spherical layer. Equations of normal perturbations in a stationary fluid are [1]

$$-\lambda_0 \mathbf{U}_0 + \nabla p + \operatorname{rot}\operatorname{rot}\mathbf{u}_0 = 0, \qquad \operatorname{div}\mathbf{u}_0 = 0, \qquad \mathbf{u}_0|_{s_1, s_2} = 0 \tag{2.1}$$

Eqs. (2.1) are self-adjoint, due to all λ being real. It is easy to show [1] that they are also positive. Therefore, all perturbations in a stationary fluid decay monotonously. It is convenient to look for the expression of $\mathbf{u}_0$ in the form of an expansion of spherical vector functions [2 and 5]

$$\mathbf{u}_0 = f(r)\,\mathbf{r}_0 Y + g(r)\, r\nabla Y + h(r)\,\mathbf{r} \times \nabla Y, \quad p = q(r)\, Y \tag{2.2}$$

Here, $Y \equiv Y_l(\vartheta) = P_l(\cos\vartheta)$ are Legendre spherical functions. In a stationary spherical layer there are two types of perturbations: azimuthal

$$\boldsymbol{\mu}_0 = h(r)\,\mathbf{r} \times \nabla Y \tag{2.3}$$

when fluid particles do not move out of their layer, and meridional

$$\mathbf{v}_s = f(r)\,\mathbf{r}_0 Y + g(r)\, r\nabla Y \tag{2.4}$$

when particles have no azimuthal velocity components. Functions f, g, h are expressed in terms of Bessel's functions of the first kind of half-integer index, and are given in [2]. For the determination of decrements μ_0 we obtain Eq.(*)

$$I_{l+1/2}(\sqrt{\mu_0})\, I_{-l-1/2}(\sqrt{\mu_0}\, a) = I_{-l-1/2}(\sqrt{\mu_0})\, I_{l+1/2}(\sqrt{\mu_0} a) \tag{2.5}$$

and for determination of decrements ν_0 Eq.

$$\frac{4}{\pi}(-1)^l \frac{2l+1}{\nu_0} - a^{l+3/2}\left[I_{l+3/2}(\sqrt{\nu_0} a)\, I_{-l+1/2}(\sqrt{\nu_0}) - I_{-l-3/2}(\sqrt{\nu_0} a)\, I_{l-1/2}(\sqrt{\nu_0})\right] -$$
$$- a^{-l+1/2}\left[I_{-l+1/2}(\sqrt{\nu_0}\, a)\, I_{l+3/2}(\sqrt{\nu_0}) - I_{l-1/2}(\sqrt{\nu_0} a) I_{-l-3/2}(\sqrt{\nu_0})\right] = 0 \tag{2.6}$$

(Decrement corrections are shown with a factor of 10^4)

a	μ_1^1		μ_2^1		μ_3^1		μ_4^1		μ_5^1	
1.7	21.3	—188	23.5	—189	26.9	—195	21.3	—202	36.9	—210
2.0	10.8	3.06	12.6	—1.64	15.4	—11.2	19.0	—15.0	23.4	—36.9
2.5	5.09	6.40	6.47	—3.46	8.48	—18.3	11.1	—35.0	14.2	—55.1
	μ_6^1		μ_7^1		μ_8^1		μ_9^1		μ_{10}^1	
1.7	43.5	—221	51.1	—231	59.8	—244	69.4	—255	80.0	—270
2.0	28.7	—52.1	34.6	—70.0	41.3	—108	48.7	—121	56.6	—
2.5	17.7	—	21.8	—	26.2	—	31.0	—	36.2	—
	μ_1^2		μ_2^2		μ_3^2		μ_4^2		ν_1^1	
1.7	81.7	—372	84.1	—441	87.6		92.2		79.5	339
2.0	40.5	—163	42.4	—212	45.4	—108	49.3	—63.4	38.6	135
2.5	18.3	—129	19.9	—153	22.2	—74.6	25.3	—37.6	16.9	98.8
	ν_2^1		ν_3^1		ν_5^1		ν_5^1		ν_6^1	
1.7	77.8	345	76.2	213	75.4	161	75.9	145	78.2	141
2.0	37.5	234	36.9	167	37.3	98.4	39.1	87.8	42.3	91.2
2.5	16.4	201	16.5	118	17.7	98.2	19.9	—	23.0	—

Equations (2.5) and (2.6) yield for each l an infinite sequence of decrements $\nu_{(0)l}^{\,i}$ and $\mu_{(0)l}^{\,i}$, with $i = 1, 2, \ldots;\ l = 1, 2, \ldots$

Eqs. (2.5) and (2.6) were solved numerically. Values of twenty lower decrements for $a = 2.5$, 2.0, 1.7 are shown in the Table above.

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It is interesting to examine the decrement behavior at large values of a, as well as for a close to unity. For simplicity's sake we shall consider the minimal roots of Eqs. (2.5) and (2.6) for $l = 1$.

A. For large values of a Eqs. (2.5) and (2.6) may be expressed respectively by

$$x = \operatorname{tg} x, \qquad x = \sqrt{\mu_0}\, a \tag{2.7}$$

$$\left(\frac{1}{x} - \frac{x}{3}\right) \operatorname{tg} x = 1, \qquad x = \sqrt{\nu_0} a \tag{2.8}$$

From (2.7) and (2.8) we easily establish that

$$\mu_0 \approx \frac{9\pi^2}{4a^2}, \qquad \nu_0 \approx \frac{\pi^2}{16a^2} \tag{2.9}$$

Thus, for large values of a the decrements of azimuthal perturbations μ_0 will be greater than those of meridional perturbations ν_0. It follows from Eqs. (2.8) and (2.9) that for large values of a decrements μ_0 and ν_0 tend to zero as $1/a^2$.

B. For a close to unity Eqs. (2.5) and (2.6) may be expressed respectively by

$$\sin(\sqrt{\mu_0}\ \delta a) = 0, \qquad \delta a = a - 1 \tag{2.10}$$

$${}^1/_2\, x \ \sin x + \cos x = 1, \qquad x = \sqrt{\nu_0}\ \delta a \tag{2.11}$$

Roots of these Eqs. are easily determined

$$\mu_0 \approx \frac{\pi^2}{(\delta a)^2}, \qquad \nu_0 \approx \frac{4\pi^2}{(\delta a)^2} \tag{2.12}$$

which means that with $a \to 1$ the decrements increase as $1/(\delta a)^2$, with decrements ν_0 situated about four times higher than the μ_0-decrements. This feature may be readily observed in the Table. As was shown in a general manner in [2], decrements μ_0 increase monotonously with the increase of number l. It is not possible to establish in a generalized form the behavior of decrements ν_0 in terms of changing l, but numerical computations show that these decrements have a minimum value (see Table) at a certain l, and that with increasing a this minimum is reached at smaller values of the l-number. It is obvious that, commencing from a certain value of a, the ν_0- and μ_0-decrements will behave monotonously with changging l.

3. Perturbations in a slowly rotating spherical layer. Perturbations in a rotating fluid satisfy Eqs. [1]

$$-\lambda \mathbf{u} + \nabla p + \operatorname{rot}\operatorname{rot}\mathbf{u} = -[(U\nabla)\mathbf{u} + (\mathbf{u}\nabla)U] \quad \operatorname{div}\mathbf{u} = 0, \qquad \mathbf{u}|_s = 0 \tag{3.1}$$

with normalization condition

$$\int \mathbf{v}_\beta^* \mathbf{u}_\alpha dV = \delta_{\beta\alpha}$$

where $\mathbf{v}$ is the solution of the conjugate problem.

We look for the solution in the form of a power series of R

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 R + \mathbf{u}_2 R^2 + \ldots, \qquad \lambda = \lambda_0 + \lambda_1 R + \lambda_2 R^2 + \ldots \tag{3.2}$$

From the symmetry problem with respect to changes of the sense of rotation it immediately follows that all coefficients of odd powers of R in the decrement expansion (3.2) are zeros. Real expansions (3.2) are valid up to the singular point. All $\mathbf{u}_n$ and λ_n may be determined by successive approximations. It is convenient to do this by means of the perturbation method, expanding $\mathbf{u}_n$ into series of the basic system of functions of the unperturbed problem (2.1). At slow rotation the perturbations cease to be purely azimuthal, or meridional, due to the operator in the right-hand side of (3.1). We shall, however, consider μ- and ν-perturbations, and track their decrements up to $R = 0$. It will be readily seen that there exist for the perturbations and their decrements power series of R as follows:

$$\begin{aligned}
\mathbf{v}_n{}^i &= \mathbf{v}_{(0)n}{}^i + R \sum_{m,k} a_{mn}{}^{ki} \boldsymbol{\mu}_{(0)m}{}^k + \ldots, \qquad \nu_n{}^i = \nu_{(0)n}{}^i + R^2 \nu_{(2)n}{}^i + \ldots \\
\boldsymbol{\mu}_n{}^i &= \boldsymbol{\mu}_{(0)n}{}^i + R \sum_{m,k} b_{mn}{}^{ki} \mathbf{v}_{(0)m}{}^k + \ldots, \qquad \mu_n{}^i = \mu_{(0)n}{}^i + R^2 \mu_{(2)m}{}^i + \ldots
\end{aligned} \tag{3.3}$$

For quadratic corrections of decrements we obtain Formulas

$$\nu_{(2)n}{}^i = N_{nn}{}^{ii} + \sum_{m,k} \frac{H_{mn}{}^{ik} K_{nm}{}^{ki}}{\nu_{(0)n}{}^i - \mu_{(0)m}{}^k}, \quad \mu_{(2)n}{}^i = M_{nn}{}^{ii} + \sum_{m,k} \frac{H_{nm}{}^{ik} K_{mn}{}^{ki}}{\mu_{(0)n}{}^i - \nu_{(0)m}{}^k} \tag{3.4}$$

$$M_{nn}^{ii} = \int \mu_{(0)n}^{i} [(\mathbf{U}_2\nabla)\mu_{(0)n}^{i} + (\mu_{(0)n}^{i}\nabla)\mathbf{U}_2]\,dV$$
$$H_{nm}^{ik} = \int \mu_{(0)n}^{i} [(\mathbf{U}_1\nabla)\mathbf{v}_{(0)m}^{k} + (\mathbf{v}_{(0)m}^{k}\nabla)\mathbf{U}_1]\,dV \qquad (3.5)$$

Integrals N_{nn}^{ii} and K_{nm}^{ik} are derived from the integrals of (3.5) by a substitution in the integrands of $\nu_{(0)n}^{i}$ for $\mu_{(0)n}^{i}$, and vice versa. Summation in (3.4) is carried out with respect to unperturbed levels. Due to the reciprocal orthogonality of spherical vectorial functions, integrals H_{mn}^{ik} and K_{mn}^{ik} differ from zero for $|m - n| = 1$ only. This "selection rule" makes computations considerably easier, reducing the summation of (3.4) to practically a single one. The computation of integrals (3.5) was carried out on an 'Argats' electronic computer at the Computing Center of the Perm' University. The first two terms of decrement expansion (3.3) computed from Eqs. (2.5) and (2.6) with the aid of Formulas (3.4) are shown in the Table.

Corrections of decrements $\mu_1^{1}, \nu_1^{1}, \nu_2^{1}$ had been computed in paper [2]. Values of corrections given in [2] do not coincide with the corresponding values shown in the Table, and corrections $\nu_{(2)2}^{1}$ and $\mu_{(2)1}^{1}$ differ also as to their signs. This lack of correlation may possibly be due to a lower accuracy of computations in [2].

4. Decrement intersection in a spherical cavity. Due to the boundary value problem (3.1) not being self-adjoint, the real expansions (3.2) are only valid up to the singular point R_*; complex decrements appear in the λ-spectrum for $R > R_*$, which indicates the onset of oscillatory perturbations of a frequency $\omega = \mathrm{Im}\,(\lambda)$. This peculiarity had already been brought to light in a number of problems (see, for example, [7 to 9]).

We shall derive the condition necessary for the onset of oscillatory perturbations. The conjugate perturbation Eqs. have the form [1]

$$-\lambda^*\mathbf{v} + \nabla q + \mathrm{rot}\ \mathrm{rot}\,\mathbf{v} - (\mathbf{U}\nabla)\mathbf{v} + \nabla(\mathbf{U}\mathbf{v}) = 0$$
$$\mathrm{div}\,\mathbf{v} = 0, \qquad \mathbf{v}_s = 0 \qquad (4.1)$$

Solutions of the basic (3.1), and of the conjugate (4.1) problems may be divided into azimuthal, (μ), and meridional (ν) parts

$$\mathbf{u} = \mathbf{u}_\mu + \mathbf{u}_\nu, \qquad \mathbf{v} = \mathbf{v}_\mu + \mathbf{v}_\nu \qquad (4.2)$$

Multiplying (3.1) in turn by $\mathbf{v}_\mu$ and $\mathbf{v}_\nu$, and (4.1) by $\mathbf{u}_\mu$ and $\mathbf{u}_\nu$, and integrating over the volume of the cavity, we obtain four integral relationships, from which we derive

$$(\lambda^* - \lambda) I \equiv (\lambda^* - \lambda) \int (\mathbf{u}_\nu \mathbf{v}_\nu - \mathbf{u}_\mu \mathbf{v}_\mu)\,dV = 0 \qquad (4.3)$$

The condition necessary for the appearance of oscillatory perturbations ($\lambda^* \neq \lambda$) is the vanishing of integral I in (4.3). As long as expansions (3.2) hold, the decrements ($\lambda^* = \lambda$) are real, with integral I differing from zero, and having different signs for the μ-, and ν-perturbations. In fact, for $R = 0$ we have $I = +1$ for the ν-perturbations, and for the μ-perturbations $I = -1$. Integral I can only vanish at the finite Reynolds number R_*. For $R = R_*$ the merger of two real decrements takes place, while for $R > R_*$ these decrements are transformed into a pair of complex conjugate decrements. In order to elucidate the behavior in the singular point neighborhood the approximate method, used in the analysis of molecular therms intersection ([8], Section. 79) may be resorted to. This method had already been used for similar purposes in papers [7 and 8].

We introduce in (3.1) the notation

$$H(R)\,\mathbf{u} \equiv (\mathbf{U}\nabla)\mathbf{u} + (\mathbf{u}\nabla)\mathbf{U} \qquad (4.4)$$

and write down (4.4) for point $R = R_0 + \delta R$ at which

$$H(R) = H(R_0) + (\partial H/\partial R)_{R_0}\delta R \qquad (4.5)$$

Let two real decrements λ_1 and λ_2 be close to each other at point R_0. We denote by $\mathbf{u}_1$ and $\mathbf{u}_2$ the solutions corresponding to λ_1 and λ_2, and by $\mathbf{v}_1$ and $\mathbf{v}_2$ their conjugate solutions. We shall look for solution $\mathbf{u}$ in the form

$$\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 \qquad (4.6)$$

Substituting Expressions (4.5) and (4.6) into (3.1), multiplying by $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$, and integrating over the cavity volume, we obtain for c_1 and c_2 a system of homogeneous linear equations.

Decrements in the neighborhood of R_0 may be derived from the compatibility condition of this system

$$\lambda = {}^1/_2\,[\lambda_1 + \lambda_2 + (V_{11} + V_{22})\,\delta R] \pm$$

$$\pm \sqrt{{}^1/_4\,[\lambda_1 + \lambda_2 + (V_{11} - V_{22})\,\delta R]^2 + V_{12}V_{21}\,(\delta R)^2} \qquad (4.7)$$

Here

$$V_{mn} = \int \mathbf{v}_m{}^* \left(\frac{\partial H}{\partial R}\right) \mathbf{u}_n dV \qquad (4.8)$$

Magnitude V_{mn} depends on the spatial symmetry of perturbations. Expansions (3.3) show that the μ- and ν-perturbations have different reflection properties in the equatorial plane. These properties of symmetry are independent of the number of expansion terms, and remain such for any R. It is readily seen that the μ_l-perturbations are symmetric for odd values of l, and antisymmetric for even values of the latter. At the same time, the ν_l-perturbations will be symmetric for even values of l, and antisymmetric when the latter are odd. Matrix elements V_{mn} of perturbations of different symmetry are equal to zero. It will be seen from Formula (4.7) that for any R_0 the identity conversion of product $V_{12}\,V_{21}$ to zero leads to a simple intersection of λ_1 and λ_2. Such intersections without the occurance of singular points are possible between μ-perturbation decrements for $\Delta l = \pm 1, \pm 3, \ldots$, between ν-perturbation decrements for $\Delta l = \pm 1, \pm 3, \ldots$, and also at intersections of the μ- and ν-perturbations decrements when $\Delta l = \pm 2, \pm 4, \ldots$. For perturbations of like symmetry the matrix elements differ from zero.

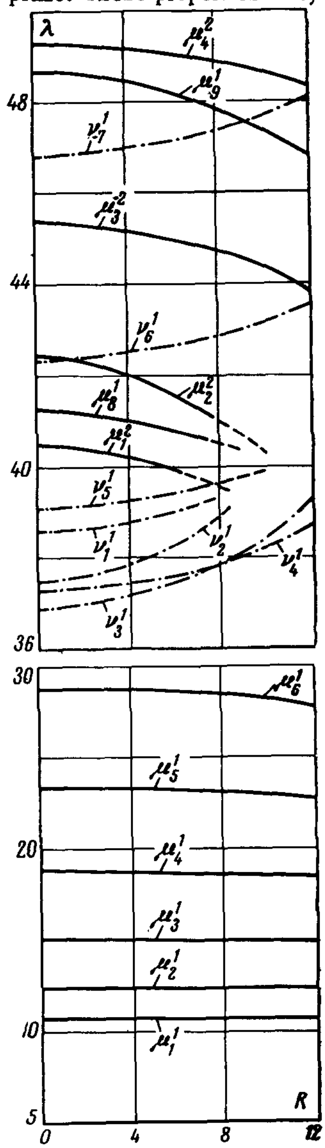

Fig. 1

It is not possible to determine in a generalized form the value and the sign of product $V_{12}V_{21}$, due to the basic flow having been defined with an approximation of the order of R^2 only. An intersection of decrements is not possible when $V_{12}V_{21} > 0$, while with $V_{12}V_{21} < 0$ and perturbations of like symmetry, but of different kind, merging of two real decrements accompanied by the formation of a pair of complex adjoints may occur.

The spectrum of lower decrements for $a = 2$ is shown in Fig. 1, where the μ-perturbation decrements are seen to decrease (with the exception of decrement $\mu_1{}^1$), while those of the ν-perturbations increase. The same behavior is also observed in the case of $a = 2.5$ (see Table). For $a = 1.7$ the smallest decrement ($\mu_1{}^1$) decreases. A great number of the μ-perturbation decrements lies between the smallest μ- and ν-perturbation decrements, and the smaller the value of a, the greater their number.

All these perturbations remain monotonous, as long as the sign of integral I in their expressions is the same. It may, therefore, be reasonably assumed that in the flow defined by (1.1) a monotonous, rather than an oscillatory instability is more likely to occur (if altogether instability takes place). Caution must be exercised in the evaluation of the critical Reynolds number in the way this was done in [2], because in the extrapolation up to $\lambda = 0$ no account was taken of the "decrement interaction", and all intersections were assumed to be simple.

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